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ON CERTAIN IDENTITIES FOR ROGERS-RAMANUJAN CONTINUED FRACTION

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Abstract: In this paper, we establish relation between Rogers-Ramanujan continued fraction $R(q)$, $R(q^2)$, $R(q^n)$ and $R(q^{2n})$ for $n = 7, 13$ *and* 17.

Keywords and Phrases: Modular equations, Theta functions, Continued fraction, Eta functions and Rogers-Ramanujan identities.

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1. Introduction

Throughout the paper, we let $|q| < 1$ and for positive integer n, we use the standard notation

$$(a; q)_0 = 1,$$

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j),$$

and

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

The well known Rogers-Ramanujan identities are

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

$G(q)$ and $H(q)$ are known as the Rogers-Ramanujan functions.
On page 236-237 of his lost notebook [7], Ramanujan has given very interesting forty algebraic identities involving Rogers-Ramanujan functions. For example

$$H(q)G^{11}(q) - q^2 G(q)H^{11}(q) = 1 + 11qG^6(q)H^6(q),$$

$$G(q^{11})H(q) - q^2 G(q)H(q^{11}) = 1.$$

In 1921 Darling [4], proved one of the forty identities in the proceedings of London Mathematical society. About the same time Rogers [9], proved ten identities including the one proved by Darling. Watson [11], proved eight of the forty identities two of which had been previously established by Rogers. Bressoud proved in his Ph.D thesis [3], fifteen more from the list of forty identities. Biagioli [2], proved remaining nine identities from the list of 40 identities of Ramanujan by using modular forms. Berndt et al [1], have proved all of forty identities in the spirit of Ramanujan.

The Rogers-Ramanujan continued fraction is

$$R(q) = \frac{q^{1/5}}{1}\,{}_{+}\frac{q}{1}{}_{+}\,\frac{q^2}{1}\,{}_{+}\,\frac{q^3}{1}\,{}_{+}\,\frac{q^4}{1}\,{}_{+\ldots}, |q|<1,$$

and

$$R(q) = q^{1/5}\frac{H(q)}{G(q)} = q^{1/5}\frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

On page 365 of his lost notebook [7], Ramanujan wrote five identities, which show the relations between $R(q)$ and five continued fraction $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$ and $R(q^5)$. He also recorded these identities at scattered places of his notebook [6]. Rogers [9] found modular equations relating $R(q)$ with $R(q^n)$, for $n = 2, 3, 5$ *and* 11, the latter equations is not found in Ramanujan's work. In [10] Vasuki and Swamy have established the relation between $R(q)$ and $R(q^7)$. Motivated by these in this paper we established the relation between $R(q)$, $R(q^2)$, $R(q^n)$ and $R(q^{2n})$ for $n = 7, 13$ *and* 17. We close this section recalling certain definitions and identities which are required to prove our main results.

Ramanujan general theta function is defined as

$$f(-a,-b) = \sum_{n=-\infty}^{\infty} (-1)^n a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (a;ab)_\infty (b;ab)_\infty (ab,ab)_\infty, \qquad |ab| < 1. \tag{1.1}$$

Ramanujan also defines special cases of $f(-a,-b)$ by

$$\phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{1.2}$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

and

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = (q;q)_\infty. \tag{1.4}$$

He also defines

$$\chi(-q) = (q;q^2)_\infty. \tag{1.5}$$

For convenience, denote $f(-q^n) := f_n$

We require the following identities found in [1, 8],

$$G^2(q)H(q^2) - H^2(q)G(q^2) = 2qH(q)H^2(q^2)\frac{f_{10}^2}{f_5^2}, \tag{1.6}$$

$$G^2(q)H(q^2) + H^2(q)G(q^2) = 2G(q)G^2(q^2)\frac{f_{10}^2}{f_5^2}, \tag{1.7}$$

$$G(q^7)H(q^2) - qG(q^2)H(q^7) = \frac{\chi(-q)}{\chi(-q^7)}, \tag{1.8}$$

$$G(q)G(q^{14}) + q^3 H(q)H(q^{14}) = \frac{\chi(-q^7)}{\chi(-q)}, \tag{1.9}$$

$$G(q^2)G(q^{13}) + q^3 H(q^2)H(q^{13}) = \sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}}, \tag{1.10}$$

$$G(q^{26})H(q) - q^5 G(q)H(q^{26}) = \sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}}, \tag{1.11}$$

$$\frac{G(q^{17})H(q^2) - q^3 G(q^2)H(q^{17})}{G(q)G(q^{34}) + q^7 H(q)H(q^{34})} = \frac{\chi(-q)}{\chi(-q^{17})}. \tag{1.12}$$

We also require

$$\frac{\phi^2(-q)}{\phi^2(-q^5)} = \frac{1-4k-k^2}{1-k^2}, \tag{1.13}$$

$$\frac{\psi^2(q)}{q\psi^2(q^5)} = \frac{1+k-k^2}{k}. \tag{1.15}$$

For a proof see [5].

2. Proof of Theorems

Theorem 2.1. *If*

$$x = R(q) \quad and \quad y = R(q^2),$$

then

$$x^3y^2 + xy^3 + x^2 - y = 0$$

Proof. From (1.6) and (1.7), it follows that

$$\frac{1 - \frac{R^2(q)}{R(q^2)}}{1 + \frac{R^2(q)}{R(q^2)}} = R(q)R^2(q^2), \tag{2.1}$$

or

$$\frac{y - x^2}{y + x^2} = xy^2 \tag{2.2}$$

This implies ,

$$x^3y^2 + xy^3 + x^2 - y = 0.$$

This complete the proof.

Theorem 2.2. *If*

$$x = R(q), y = R^2(q^2), z = R(q^7) \quad and \quad w = R^2(q^{14}),$$

then

$$\frac{(y-z)^3(zw+\frac{1}{w})^3}{(wx+1)^3(xy+\frac{1}{y})^3} = \frac{(-x^2y^2-4xy+1)(xy)(-w^2z^2+1)(-w^2z^2+wz+1)}{(-x^2y^2+1)(-x^2y^2+xy+1)(-w^2z^2-4wz+1)(zw)}$$

Proof. From (1.8) and (1.9), it follows that

$$\left(\frac{R_2-R_7}{1+R_1R_{14}}\right) = \frac{\chi^2(-q)}{\chi^2(-q^7)}\left(\frac{G_1}{G_2}\right)\left(\frac{G_{14}}{G_7}\right) \tag{2.3}$$

Using (1.13) and (1.14) and raising the power to three on both sides and simplifying, we obtain

$$\left(\frac{y-z}{1+xw}\right)^3\left(\frac{zw+\frac{1}{w}}{xy+\frac{1}{y}}\right)^3 - \left(\frac{1-4xy-x^2y^2}{1-x^2y^2}\right)\left(\frac{xy}{1+xy-x^2y^2}\right)$$
$$\left(\frac{1-z^2w^2}{1-4zw-z^2w^2}\right)\left(\frac{1+zw-z^2w^2}{zw}\right) = 0 \tag{2.4}$$

This implies,

$$\frac{(y-z)^3(zw+\frac{1}{w})^3}{(wx+1)^3(xy+\frac{1}{y})^3} - \frac{(-x^2y^2-4xy+1)(xy)(-w^2z^2+1)(-w^2z^2+wz+1)}{(-x^2y^2+1)(-x^2y^2+xy+1)(-w^2z^2-4wz+1)(zw)} = 0.$$

This complete the proof.

Theorem 2.3. *If*

$$x = R(q), y = R^2(q^2), z = R(q^{13}) \quad and \quad w = R^2(q^{26}),$$

then

$$\frac{(yz+1)^{12}(zw+\frac{1}{w})^{12}}{(x-w)^{12}(xy+\frac{1}{y})^{12}}$$
$$= \frac{(-w^4z^2-4w^2z+1)(-w^4z^2+1)^5(-x^2y^4-xy^2+1)(x^5y^{10})}{(z^5w^{10})(-x^2y^4+1)^5(-w^4z^2+z+1)(-x^2y^4-4xy^2+1)}.$$

Proof. From (1.10) and (1.11), it follows that

$$\left(\frac{1+R_2R_{13}}{R_1-R_{26}}\right)\left(\frac{R_{13}R_{26}+\frac{1}{R_{26}}}{R_1R_2+\frac{1}{R_2}}\right)=\frac{\chi^2(-q^{65})}{\chi^2(-q^5)} \tag{2.5}$$

Using (1.13) and (1.14) and raising the power to twenty four on both sides and simplifying, we obtain

$$\left(\frac{1+yz}{x-w}\right)^{24}\left(\frac{zw+\frac{1}{w}}{xy+\frac{1}{y}}\right)^{24}-\left(\left(\frac{1-4zw^2-z^2w^4}{1+z-z^2w^4}\right)\left(\frac{1-z^2w^4}{zw^2}\right)^5\right)^2 \left(\left(\frac{1-xy^2-x^2y^4}{1-4xy^2-x^2y^4}\right)\left(\frac{xy^2}{1-x^2y^4}\right)^5\right)^2=0 \tag{2.6}$$

This implies,

$$\frac{(yz+1)^{12}(zw+\frac{1}{w})^{12}}{(x-w)^{12}(xy+\frac{1}{y})^{12}} - \frac{(-w^4z^2-4w^2z+1)(-w^4z^2+1)^5(-x^2y^4-xy^2+1)(x^5y^{10})}{(z^5w^{10})(-x^2y^4+1)^5(-w^4z^2+z+1)(-x^2y^4-4xy^2+1)}=0.$$

This complete the proof.

Theorem 2.4. *If*

$$x=R(q), y=R^2(q^2), z=R(q^{17}) \quad and \quad w=R^2(q^{34}),$$

then

$$\frac{(y-z)^{24}(zw+\frac{1}{w})^{24}}{(wx+1)^{24}(xy+\frac{1}{y})^{24}}= \left(\frac{(-x^2y^2-4xy+1)(x^6y^{11})(-w^2z^2+1)(-w^2z^2+wz+1)(-x^2y^4+xy^2+1)}{(-x^2y^4+1)^5(z^6w^{11})(-x^2y^2+1)(-x^2y^2+xy+1)(-w^2z^2-4wz+1)}\right) \times\left(\frac{(-w^4z^2+1)^5(-4w^6z^3+1)}{(-x^2y^4-4xy^2+1)(-w^4z^2+w^2z+1)}\right)$$

Proof. From (1.12), it follows that

$$\left(\frac{R_2-R_{17}}{1+R_1R_{34}}\right)\left(\frac{R_{17}R_{34}+\frac{1}{R_{34}}}{R_1R_2+\frac{1}{R_2}}\right)-\frac{\chi(-q)}{\chi(-q^5)}\frac{\chi(-q^{85})}{\chi(-q^5)}\frac{\chi(-q^{85})}{\chi(-q^{17})}=0 \tag{2.7}$$

Using (1.13) and (1.14) and raising the power to twenty four on both sides and simplifying, we obtain

$$\left(\frac{y-z}{1+xw}\right)^{24}\left(\frac{zw+\frac{1}{w}}{xy+\frac{1}{y}}\right)^{24}-\left(\frac{1-4xy-x^2y^2}{1-x^2y^2}\right)\left(\frac{xy}{1+xy-x^2y^2}\right)$$
$$\left(\frac{1-z^2w^2}{1-4zw-z^2w^2}\right)\left(\frac{1+zw-z^2w^2}{zw}\right)\left(\frac{xy^2}{1-x^2y^4}\right)^5\left(\frac{1+xy^2-x^2y^4}{1-4xy^2-x^2y^4}\right)$$
$$\left(\frac{1-z^2w^4}{zw^2}\right)^5\left(\frac{1-4zw^2-z^2w^4}{1+zw^2-z^2w^4}\right)=0 \quad (2.8)$$

This implies,

$$\frac{(y-z)^{24}(zw+\frac{1}{w})^{24}}{(wx+1)^{24}(xy+\frac{1}{y})^{24}}-$$
$$\left(\frac{(-x^2y^2-4xy+1)(x^6y^{11})(-w^2z^2+1)(-w^2z^2+wz+1)(-x^2y^4+xy^2+1)}{(-x^2y^4+1)^5(z^6w^{11})(-x^2y^2+1)(-x^2y^2+xy+1)(-w^2z^2-4wz+1)}\right)$$
$$\times\left(\frac{(-w^4z^2+1)^5(-4w^6z^3+1)}{(-x^2y^4-4xy^2+1)(-w^4z^2+w^2z+1)}\right)=0.$$

This complete the proof.

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